Identify and avoid the risks involved in using visual and hands-on representations, and use these suggestions to scaffold conceptual congruence.
Representations that create informative visual displays are powerful tools for communicating mathematical concepts. The National Council of Teachers of Mathematics encourages the use of manipulatives (NCTM 2000). Manipulative materials are often used to present initial representations of basic numerical principles to young children, and it is through these early developmental experiences that children frequently receive their first introduction to formal mathematics. Manipulative displays are well suited to serve as proxies for real-world problems, taking on the role of representing quantities. Teachers intuitively assemble manipulative displays to construct such representations, often attaching language to scaffold the real-world connections they are trying to portray. Many children prosper from the interaction; however, others do not. For the latter, confusion can result when the arrangement of the display is not clearly connected to the concept being taught (Garcia-Mila, Marti, and Teberosky 2004).
The aims of this article are to—

1. identify the rationale for using manipulative displays as tools for communicating concepts;
2. describe the desired attributes teachers should consider when selecting and implementing manipulatives;
3. define conceptual congruence; and
4. offer some suggestions for helping teachers achieve greater congruence between the numerical concepts and procedures they are teaching and the manipulative displays they are using to represent them.

We also present an example of a $1 \times 10$ ten-frame display to use as a tool to facilitate conceptual congruence.

**Rationale for selecting and using manipulative displays**
Experts acknowledge the instructional power of using manipulative displays to convey important mathematical concepts (e.g., Fuson 2009; McNeil and Jarvin 2007; NCTM 2000; Swan and Marshall 2010). First, we examine teachers’ use of manipulatives as young children transition from play-based, real-world knowledge of quantities to a formal base-ten numeral system.

**Number sense**
The acquisition of number sense is gradual for young learners, and manipulatives are often used as a tool to help children make the connection between groups of objects as quantities and the representation of these quantities in the base-ten numeral system (Boggan, Harper, and Whitmire 2010; Fuson et al. 2000; Kosko and Wilkins 2010).

**Cardinality**
Teachers frequently use manipulative displays during critical stages throughout the transition to formal mathematics. First and foremost is the acquisition of the cardinality principle (Geary, Bow-Thomas, and Yao 1992; Morin and Franks 2009). Cardinality is the last number stated when a group of objects is counted. By learning cardinality, children learn that the elements of a group (e.g., rabbits, marbles) can now be identified by a quantity (e.g., *the group of five*) in addition to the class name (Morin and Franks 2009).

Children who are just beginning to acquire cardinality are often asked to count a set of objects and attach a summative phrase, such as *There are six rabbits altogether*. These manipulative displays usually involve a homogeneous grouping of familiar objects (e.g., candies, toys). Eventually, when children begin their formal education, teachers introduce objects that are more standardized (e.g., disks, dry noodles, bundles of rods to represent units of ten) that are used as tangible representations of quantifiable commodities encountered in real life (e.g., years, distances, weights) (Uttal, Scudder, and DeLoache 1997). As the children progress with their learning, these tangible counters are gradually replaced by such semiconcrete counters as tally marks and dots. Throughout this transition, teachers scaffold implicit and explicit associations with number names and the base-ten numerals that represent them. The strength of these associations among concrete objects, semiconcrete counters, number names, numerical symbols, and the child’s understanding of the mathematical concept is strongly influenced by the attributes of the manipulative display and the way that teachers arrange and present that manipulative display.
Effective manipulative displays
A judicious approach to selecting appropriate manipulatives is important. Certainly the physical properties of size, shape, and texture are important along with other considerations, such as storage. However, conceptual attributes are probably more important.

For younger learners, manipulatives can be chosen for their resemblance to the content of the problems that are to be solved. For example, if the story problem is about apples and oranges, small laminated pictures of apples and oranges could be used to illustrate the story. Later however, these content-oriented manipulatives must be replaced with semi-concrete representations of the content (Fuson 2009). At this stage, manipulatives that come in contrasting colors will be useful. Thus, red disks could be used to represent apples, and orange disks could be used to represent oranges. As teachers shift to manipulative displays that are more abstract, a final conceptually relevant consideration should be to use a structured ten-frame template to organize the manipulatives (Clements 1999).

Benefits and cautions
Using manipulative materials to augment instruction requires considerable forethought that belies the apparent simplicity of the task. After all, it is only counting, isn’t it? Well, it is not quite that simple. A lot can go wrong within a lesson that relies on concrete manipulative displays if a teacher does not maintain firm instructional control of the materials. Improperly selected, arranged, and presented, manipulative displays can compromise the conceptual congruence between the display and the targeted math concept. For the purposes of this discussion, we define adequate conceptual congruence as—

the ability of the teacher to use a manipulative display to scaffold an accurate answer, in the most economical way possible, while preserving the least ambiguous representation of the mathematical concept.

The latter part of this statement is crucially important for helping children comprehend key mathematical concepts, such as joining amounts (i.e., addition) and breaking apart amounts (i.e., subtraction).

Anyone who has tried to teach a mathematical concept to a young child appreciates that the use of manipulatives can be distracting (e.g., their appearance, their use for alternative purposes like toys). Although the potential for unwanted distraction is high, it is unlikely that this alone would be enough to dissuade good teachers from employing manipulative materials in their classrooms. Setting aside the risk of unwanted distraction, however, other important considerations exist that may have graver consequences than distractibility alone. Improperly displayed manipulatives threaten the conceptual continuity implied by the concrete manipulations of the materials. This possibility can occur with any learner but is especially important for teachers who have learners who are developmentally immature or have not been taught the proper use of manipulatives (Fuson et al. 2000). A discussion of these conceptual pitfalls follows.

Transitioning to semiconcrete manipulatives
Uttal, Scudder, and DeLoache (1997) argued that so-called concrete manipulatives are actually symbolic representations of concepts. For students to use manipulatives effectively, they must be able to view the manipulatives (e.g., blocks) both as symbolic representations of quantities and as objects in their own right (i.e., cubes for building towers and houses). As Fuson (2009) suggests, it is advantageous for early mathematics instruction to progress from the use of concrete manipulatives to representations of these manipulatives that are more abstract, by using dots or tally marks on paper, arranged in ways that parallel the concrete displays. Thus, explicitly instructing students in the use of manipulatives as symbolic representations is important for teachers to do (Swan and Marshall 2010; Yanzick and Samelson 2011). The time taken to transition from concrete to semiconcrete manipulatives depends on students’ prior learning experiences and developmental maturity. Untaught or developmentally immature learners need more time with concrete manipulatives than typical learners (Samelson 2009).
Transitioning to independent work
Making manipulatives available while students are solving problems on their own does not guarantee that children will use these materials appropriately. The transition to independent work is also important. Despite the fact that first graders with low oral language skills eagerly made use of manipulative displays, Humbert and Samelson (2010) found that these students employed a variety of inaccurate solution strategies while solving basic arithmetic word problems independently. These inaccuracies included student errors in counting, errors in assigning the appropriate quantity to one of the characters in the problem, and response strategies where the child simply chose one of the quantities in the problem as his or her answer (Hudson 1983). Some of these faulty solution strategies were avoided by providing explicit instruction in the use of manipulative displays to support problem comprehension. This instruction helped achieve conceptual congruence between the manipulative display and the word problem schema (Yanzick and Samelson 2011).

Achieving conceptual congruence
What follows is a series of examples that illustrate possible threats to the conceptual congruence of mathematical representations that rely on manipulative displays, with implications for how teachers can carefully create manipulative displays. With regard to counting accuracy, teachers should model think-alouds (Fogelberg et al. 2008) of their manipulative counts, to correct for a faulty correspondence between the touched object and the oral count, the double touch of one object, the faulty discrimination between the counted objects and the uncounted ones; or the migration of one manipulative into another group so that it does not get counted with the intended group. All these occurrences compete with cardinality, solution accuracy, and conceptual congruence, especially in children who are untaught or developmentally immature. These children are often oblivious to their miscounts and miscalculations, and thereby are unable to reach congruence between the manipulative display and the concept being presented; whereas typical learners often sense that their count or answer does not match the number of objects or the schema of the problem (Samelson 2009). Modeling solution strategies increases children’s awareness of the congruence (or lack thereof) between their count and the display (Fogelberg et al. 2008).

To achieve greater conceptual congruence, learners are also better served when teachers select manipulative displays that emphasize the quantitative dimensions of the manipulatives (e.g., cardinality) rather than their physical characteristics (see fig. 1). We can see that the physical difference between the disks and the cubes provides an incongruent image for the child because the disks are somewhat larger (i.e., the quantitatively smaller set seems larger). This lack of congruence often competes with the estimate of each group’s numerosity (Babai 2010; Feigenson, Carey, and Spelke 2002). Piaget (1965) and others (e.g., Fuson 2009; Uttal, Scudder, and DeLoache 1997; Willoughby 1988) have observed such confusion in some children’s performance on similar tasks. Counting each group separately might facilitate a response to the question Which group has the larger amount? To respond accurately to the question How much more is one quantity than the other? however, the child must have sufficient number sense to discriminate the difference in magnitude between the quantities. A visually incongruent manipulative display does not sufficiently support an

FIGURE 1
In this cardinal number comparison task, students count two groups of objects to estimate which group is larger.
accurate solution to this question; therefore, focusing the children’s attention specifically on the attribute of cardinality rather than on the objects themselves is important for the teacher to do. Unless children are cognitively ready to conceptualize and compare quantities and then are taught to use an efficient strategy (e.g., for fig. 1, a pairing-off strategy by pairing one disk to one cube and counting the leftovers), they may continue to encounter difficulty. Next it is important that teachers guide children to make a developmentally appropriate transition from strategies that involve only perceptual awareness of the physical components of the count (e.g., pairing-off strategies) to strategies that purposefully rely on the conceptual features of the base-ten numbering system (e.g., a ten-frame template).

As observed by Hudson (1983), children new to comparison problems often answer the question How much more? by restating the larger amount (i.e., “seven”). Restating the question with more explicitness might provide an intermediate support for the moment (e.g., How many cubes will be left if we pair the disks with the cubes?) (Hudson 1983), but the more enduring goal of laying the foundation for computational concepts will be quite limited with this approach, as this type of strategy may not generalize well to numeral-based computation (e.g., \(7 - 5 = \square\)), because twelve objects are still on display (i.e., five disks plus seven cubes). How does the child grasp the concept of subtracting the 5 from the 7 when the 5 does not appear to be part of the 7 in the display but is in fact in addition to the 7 and separate from it? One solution is to use a ten-frame template in conjunction with the display. In addition to conserving the base-ten concept, ten-frame templates also provide an accessible structure for the learner to cement the concept and render an accurate solution to the How much more? question. An example of a 1 × 10 ten-frame is introduced in the following section.

**Ten-frame templates**

As children transition to manipulatives that are increasingly abstract, teachers should consider displays that offer some efficiency in terms of counting, while maintaining conceptual congruence. Teachers are encouraged to consider the use of a base-ten template where abstract manipulative materials can be positioned to represent quantities. A template functions much like an egg carton. When an egg or two are removed from an egg carton, the analogue image of what constitutes a dozen makes the calculation—of how many eggs remain and how many are missing—instant and effortless, thereby eliminating the need for counting. Contrast this to a bowl of ten eggs where the count is obscured by the random arrangement of the eggs. The bowl does not render the same numerical advantage as the egg carton, and the analogue image of a dozen is lost. So, ten-frame templates that preserve the ten-ness of the base-ten structure provide an analogue advantage insofar as the relationship of a particular quantity to ten is always in full view. The young learner who has acquired this analogue image is at a distinct advantage when subsequently introduced to such procedures as regrouping.

Another attribute of a conceptually congruent base-ten template is the ease with which it can facilitate subitizing (i.e., determining the cardinal number of a set without having to count). For example, figure 2 shows a contiguous ten-frame with five slots shaded and another five slots unshaded. This 1 × 10 ten-frame represents a slight, but conceptually
An important modification to the more typical 2 x 5 ten-frame orientation promoted by Clements (1999). The essential feature of the 1 x 10 design is the ability of the template to manage the count more precisely than the 2 x 5 template. By standardizing the orientation of the template with the shaded portion closest to the child, the child can learn to map the template to the numerical sequence 1–10 rather than two sets of five. For example, the first frame in the shaded section would always be the value 1, and the second frame, the value 2, and so on. The first frame in the unshaded area would always be 6, and the second one would always be 7, and so forth. With effective instruction, children can learn the map, thereby increasing the probability that they can determine how many manipulatives are displayed without counting (i.e., subitizing) (Clements 1999). Working with manipulative displays that have contrasting colors also facilitates conceptual congruence. For example, figure 2 clearly shows that five combined with two more is seven. It also shows that when you have a group of seven, you can break it apart into a group of five and a group of two to facilitate a demonstration of subtraction. The display generalizes well to cover not only take-away situations but also comparison problems (e.g., How much more?) and classification problems (e.g., How many apples are in the basket of fruit?). Another critical attribute of the template is the feature that the value of ten is constant regardless of the quantity you are displaying. Thus, with the seven disks, it is easy to see that three more would be needed to make a full ten. It is essential that children understand this concept before they learn calculation skills that involve regrouping. As children become more advanced, the template could be used to demonstrate start-unknown and gain-unknown problems (i.e., missing addends), a prerequisite for algebraic thinking. For example, with 5 + □ = 7, placing five counters on the template and a finger on the seven-frame, the child counts the vacant frames up to the seventh for a count of two.
The advantages above of the $1 \times 10$ template (see fig. 2) may be slightly less accessible to the learner using the more traditional $2 \times 5$ ten-frame template. For example, figure 3 shows a $4 + 3$ problem and a $7 - 3$ or a $7 - 4$ problem using a $2 \times 5$ ten-frame. You can see the break in the continuity of the three as two of the counters that represent the three are separated from it because they are positioned in the second row. This may compete with the learner’s understanding of the relationship of the three as a subset of seven. The advantage of the $1 \times 10$ over the $2 \times 5$ ten-frames in terms of conserving conceptual congruence may be small for most typical learners but important for untaught or developmentally immature learners.

**Focusing on achieving greater congruence**

Achieving congruence between numerical concepts and manipulative displays is critical for all students, especially those students who may be at risk for failure resulting from the restrictions imposed by poverty, cultural difference, or disability (Hudson and Miller 2006). Congruence between numerical concepts and manipulative displays helps children establish critical numerical concepts and provides them with a visual representation that is more coherent and to which conceptual and procedural language can be attached. This congruence facilitates the transition from concrete to formal abstract mathematical knowledge. Incongruent manipulative displays can confuse children and render them less prepared to learn at a level that is commensurate with their potential. We recommend that teachers focus on achieving greater congruence between the numerical concepts and procedures they are teaching and the manipulative displays they are using to represent them.

**BIBLIOGRAPHY**


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